Indian Statistical Institute
First Semester 2006-2007
Mid-Semester Examination
M.Math II Year
Representation of Groups

Time: 3 hrs | Date:22-09-06 | Max. Marks : 100

Instructor: B Bagchi

1. Let $\mathcal{K}$ be a field and $\mathbf{V}$ be a finite dimensional vector space over $\mathcal{K}$. Let $\mathcal{L}$ be the $\mathcal{K}$-algebra of all linear operators from V into $\mathbf{V}$.

   (a) For any subspace $W$of $\mathbf{V}$, let $I_W = \{T \in \mathcal{L} : \ker(T) \supseteq W\}$. Show that I_W is a left ideal of $\mathcal{L}$.

   (b) If $T_1,\ T_2 \in \mathcal{L}$ with $\ker(T_1) \subseteq \ker(T_2)$ then show that there is an $S \in \mathcal{L}$ such that $T_2 = ST_1$.

   (c) For any two subspaces $W_1,\ W_2$ of $\mathbf{V}$, show that there are $T_1,\ T_2 \in \mathcal{L}$ such that $\ker(T_1) = W_1,\ \ \ker(T_2) = W_2,\ \ \ker(T_1 + T_2) = W_1 \cap W_2$.

   (d) Let I be any left ideal of $\mathcal{L}$. Pick a basis $\{T_1, \dots, T_k\}$ of I and put $W = \bigcap_{i=1}^{k} \ker(T_i)$. Then use parts (b) and (c) to show that $I = I_W$.

   (e) Let $B(V)$ be the set of all subspace of $\mathbf{V}$, partially ordered by $\subseteq$. Let $g(\mathbf{V})$ be the set of all left ideals of $\mathcal{L}$, partially ordered by $\subseteq$. Show that $W \mapsto I_W$ is an order reversing bijection between these two partially ordered sets.

   (f) Find all minimal left ideals of $\mathcal{L}$ and describe all possible decompositions of $\mathcal{L}$ as direct sums of minimal left ideals of $\mathcal{L}$.

   [5+10+10+10+5+5 = 45]

2. Let G be a finite group and $\mathcal{K}$ an algebraically closed field such that char($\mathcal{K}$) does not divide the order of G.

   (a) State and prove Schur's Lemma for $\mathcal{K}$-representations of G.

   (b) Deduce from Schur's Lemma that if G is abelian then all the irreducible $\mathcal{K}$-representations of G are of degree one.

   (c) Explicitly describe all the irreducible $\mathcal{K}$ representations of $G = Z_n$, the cyclic group of order n. [5+5+5 = 15]

3. (a) With the hypothesis as in Question no. (2), lt π be a $\mathcal{K}$-representation over a vector space $\mathbf{V}$ and put $V^G = \{v \in \mathbf{V} : \pi(g)v = v \quad \forall\, g \in G\}$ and $P = \frac{1}{\#G} \sum\limits_{g \in G} \pi(g)$. Show that $P^2 = P$ and $\operatorname{ran}(P) = V^G$. Conclude that if χ is the character of π then $\frac{1}{\#G} \sum\limits_{g \in G} \chi(g) = \dim V^G$.

   (b) Now let π be a transitive permutation representation of G. Deduce from part (a) that if $\operatorname{char}(\mathcal{K}) = 0$ then the average number of symbols fixed by elements of G is $= 1$. [10+5=15]

4. (a) With the hypotheses as in Question No. (2), show that $\operatorname{char}(\mathcal{K})$ does not divide the degree of any irreducible representation of G over $\mathcal{K}$.

   (b) If, further, $\operatorname{char}(\mathcal{K}) = 0$ then show that the degree of each irreducible $\mathcal{K}$-representation of G divides the order of G. [10+15=25]